Lecture 3 (Sec 5.5)

☆ Multiple Eigenvalue Solutions:

Let's start with a comparison
In MA 206 we solved a 2nd order linear system like:

$$y'' - 6y' + 9y = 0$$

We can convert this into a 1st order system

let $x_1 = y$ $\qquad$ $x_1' = y' = x_2$

$x_2 = y'$ $\qquad$ $x_2' = y'' = 6y' - 9y$

$\qquad = 6x_2 - 9x_1$

In vector form:

$$\underline{x}' = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \underline{x}$$

These two differential equations are equivalent.

So let's compare their solutions side by side

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	1D	2D
equation:	$y'' - 6y' + 9y = 0$	$\underline{x}' = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \underline{x}$
characteristic equation:	$r^2 - 6r + 9 = 0$	$\begin{vmatrix} -\lambda & 1 \\ -9 & 6-\lambda \end{vmatrix} = 0$ $\lambda^2 - 6\lambda + 9 = 0$
roots / eigenvalues	$r = 3$, multiplicity 2 (repeated root)	$\lambda = 3$ <u>algebraic</u> <u>multiplicity</u> $= 2$
fundamental solutions	$y_1 = e^{3t}$ Need a 2nd linearly independent solution $\rightarrow$ multiply by t $y_2 = te^{3t}$	$\lambda = 3$ $(\underline{A} - 3\underline{I})\underline{v} = 0$ $\begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $-3v_1 + v_2 = 0$ $v_2 = 3v_1$ $\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ But there's only 1 unique eigenvector <u>geometric</u> <u>multiplicity</u> $= 1$ Need to find a 2nd linearly independent vector. Call it $\underline{u}$

<u>Note</u>: When geometric multiplicity < algebraic multiplicity

we say the eigenvalue is <u>defective</u>

We have one fundamental solution

$$\underline{x}^{(1)} = e^{3t}\begin{bmatrix}1\\3\end{bmatrix}$$

Need to find a <u>generalized</u> <u>eigenvector</u> $\underline{u}$

Solve: $$(\underline{\underline{A}} - \lambda \underline{\underline{I}})^2 \underline{u} = \underline{0}$$

or: $$(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{u} = \underline{v}$$

$$\begin{bmatrix}-3 & 1\\-9 & 3\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$$

$$-3u_1 + u_2 = 1$$

$$u_2 = 1 + 3u_1$$ choose $u_1 = 1$

$$\underline{u} = \begin{bmatrix}u_1\\1+3u_1\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$$

So the second fundamental solution will be:

$$\underline{x}^{(2)} = e^{3t}\ \underline{v}t + \underline{u})$$

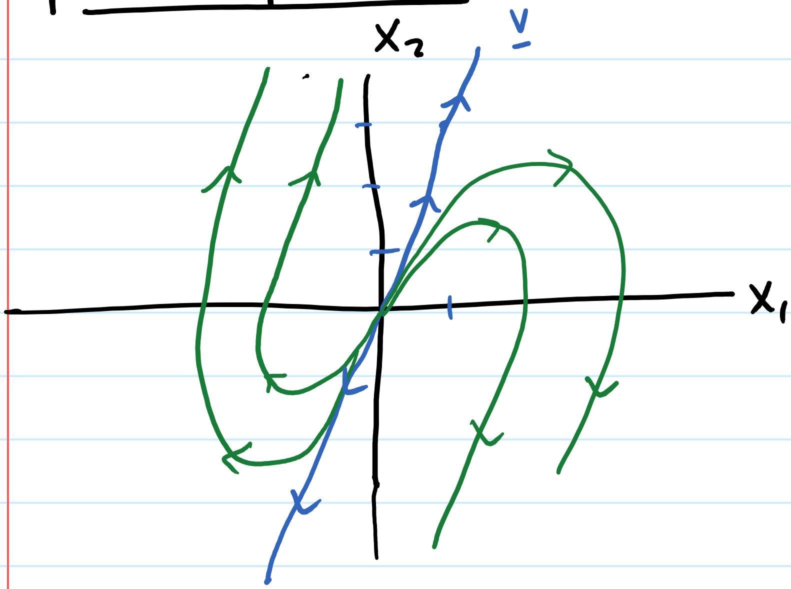
$$= e^{3t}\left(\begin{bmatrix}1\\3\end{bmatrix}t + \begin{bmatrix}1\\4\end{bmatrix}\right)$$

General solution:

$$\underline{x}(t) = c_1 e^{3t}\begin{bmatrix}1\\3\end{bmatrix} + c_2 e^{t}\left(\begin{bmatrix}1\\3\end{bmatrix}t + \begin{bmatrix}1\\4\end{bmatrix}\right)$$

Compare to: $$y(t) = c_1 e^{3t} + c_2 t e^{3t}$$

phase portrait:



1. Draw the eigenvector $\underline{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

2. $\lambda = 3 > 0$ so arrows point out

3. $t \to \infty$

$\underline{x} = e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} t$

goes parallel to $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Called an improper nodal source

Lets do a 3x3 example

Ex: $\underline{x}' = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \underline{x}$

Characteristic eqn:

$$\begin{vmatrix} 1-\lambda & 3 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)^2 + 0 + 0 = 0$$

$\lambda = 1$

$\lambda = 2$ algebraic multiplicity = 2

$\lambda_1 = 1$

$$\begin{bmatrix} 0 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} 3v_2 - v_1 = 0 \\ \rightarrow v_2 = 0 \\ \rightarrow v_3 = 0 \end{matrix}$$

So v_1 is a free variable. Choose $v_1 = 1$

$$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_2 = 2$

$$\begin{bmatrix} -1 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1 equation and 3 unknowns

$\rightarrow$ there will be 2 free variables

$$-v_1 + 3v_2 - v_3 = 0$$

$$3v_2 = v_1 + v_3$$

Choose $v_2 = 1$

$v_1 = 0$

$\rightarrow 3 = v_3$

$\rightarrow \quad \underline{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

choose $v_2 = 1$

$v_3 = 0$

$\rightarrow 3 = v_1$

$\rightarrow \quad \underline{v}^{(3)} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

Note: There are 2 linearly independent eigenvectors for $\lambda_2 = 2$

$\rightarrow$ geometric multiplicity $= 2$

The eigenvalue $\lambda = 2$ is called <u>complete</u>

General solution:

$$\underline{x}(t) = c_1 e^{t}\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}0\\1\\3\end{bmatrix} + c_3 e^{2t}\begin{bmatrix}3\\1\\6\end{bmatrix}$$